A Generalized ECG Dynamic Model with Asymmetric Gaussians and its Application in Model-based ECG Denoising

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***Abstract*— Constrained by the symmetries of its Gaussian elements, traditional ECG dynamic model has difficulty in accurately representing complicated ECG waveforms. In order to overcome to this limitation, this paper proposes a generalized EDM by introducing asymmetric Gaussians into the model instead of symmetric ones. The generalized EDM is then applied to the model-based ECG denoising framework using an extended Kalman filter (EKF). Experiments are conducted based on the MIT-BIH Arrhythmia database, and the results show that the proposed EDM is able to model a wider range of ECG morphologies than the traditional one, and consequently improves the denoising performance.**

Keywords- Electrocardiogram; dynamic model; denoising

I. INTRODUCTION

Electrocardiogram (ECG) is a recording of the bioelectric potential produced by rhythmical cardiac activities, contraction and relaxation. Obtained by a noninvasive technique, ECG has been extensively used for heart disease diagnosis in hospital, as well as patient monitoring at home, since it can provide valuable information of the heart functional conditions. However, ECG is usually contaminated by kinds of noises, for example, electromyographic (EMG) noise, which overlaps the ECG signal in the frequency domain. Therefore, traditional band-pass filtering cannot suppress such noises efficiently [1].

Based on an ECG dynamic model (EDM) previously developed for generating realistic synthetic ECG [2], Sameni *et al.* proposed a nonlinear Bayesian filtering framework using extended Kalman filter (EKF) for ECG denoising [3], and Sayadi *et al*. further modified this work by extending the previous two-dimensional EKF structure to a 17-dimensianal case [4]. Although these nonlinear filtering approaches have demonstrated desirable denoising performance, they are limited by the EDM which utilizes a few Gaussians to model one heartbeat of ECG signals and therefore has difficulty in modeling those containing asymmetric waveforms.

In this paper, asymmetric Gaussians (AGs) are introduced to generalize the ECG dynamic model (EDM) by replacing the previous symmetric ones, and the extended Kalman filter (EKF) is then utilized to filter noisy ECG signals based on the generalized EDM. Experiment results show that our method has improved the model's ability of approximating ECG morphologies, as well as the denoising performance.

The rest of this paper is organized as follows. Section II introduces the required background knowledge of the ECG dynamic model and extended Kalman filter. Details of the proposed method are presented in Section III. Section IV provides experimental results, and conclusion and future work come in Section V.

II. BACKGROUND

A. ECG Dynamic Model

A realistic synthetic ECG generator was first proposed by McSharry *et al.* [2], using a set of 3-D state equations to generate a trajectory in the Cartesian coordinates. The dynamic equations were then transformed into the polar form for a simpler compact set by Sameni *et al.* [3].

Briefly speaking, in this model, several Gaussians are utilized to approximate the feature waves (P, Q, R, S and T waves, see Fig. 1) in one heartbeat of ECG. To handle the heart rate variation, each heartbeat signal is linearly mapped into the interval [0, 2π] as follows:

$$ECG(\theta) = \sum_{i\in\{P,Q,R,S,T\}} a_i \exp\left(-\frac{(\theta-\theta_i)^2}{2b_i^2}\right) \quad (1)$$

where $\theta \in [0, 2\pi]$, a_i, b_i and θ_i respectively represent the amplitude, width and location of the *i*-th wave (or Gaussian). Using the Euler forward difference, we obtain the discrete iteration form:

$$\begin{aligned} ECG(k+1) - ECG(k) &= \delta_\theta \times \left.\frac{dECG(\theta)}{d\theta}\right|_{\theta=\theta(k)} \\ &= -\sum_{i\in\{P,Q,R,S,T\}} \delta_\theta \frac{a_i(\theta(k)-\theta_i)}{b_i^2} \exp\left(-\frac{(\theta(k)-\theta_i)^2}{2b_i^2}\right) \end{aligned} \quad (2)$$

where $\delta_\theta = 2\pi / N$ is angular displacement per sampling time, N is the beat length, and $\theta(k) = k\delta_\theta$.

As it is shown in (2), the ECG signal is modeled with a sum of five Gaussian functions, each of which is located at a

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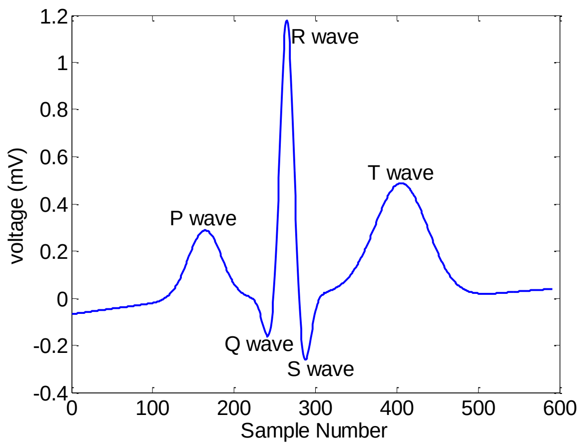


Figure 1. A typical synthetic ECG signal

specific angular position θ_i, and a typical synthetic ECG signal is illustrated in Fig. 1.

B. Extended Kalman Filter

As a nonlinear extension of the conventional Kalman filter, the extended Kalman filter (EKF) is developed for systems with nonlinear dynamic models [5]. Consider a discrete nonlinear system with the state vector x_k, input vector u_k, and observation vector y_k. The dynamic state space model can be formulated as

$$\begin{aligned} x_{k+1} &= f(x_k, u_k, w_k) \\ y_k &= g(x_k, v_k) \end{aligned} \tag{3}$$

where $f(\cdot)$ is the state evolution function, $g(\cdot)$ is the observation function, w_k and v_k stand for process and measurement noises respectively, with covariance matrices $Q_k = E\{w_k w_k^T\}$ and $R_k = E\{v_k v_k^T\}$. The initial state estimate of the state x_0 is defined as $\bar{x}_0 = E\{x_0\}$, with covariance matrix $P_0 = E\{(x_0 - \bar{x}_0)(x_0 - \bar{x}_0)^T\}$.

We need to derive a linear approximation of (3) near a desired reference point $(\hat{x}_k, u_k, \hat{w}_k, \hat{v}_k)$ so as to apply the Kalman filter formalism:

$$\begin{aligned} x_{k+1} &\approx f(\hat{x}_k, u_k, \hat{w}_k) + A_k(x_k - \hat{x}_k) + F_k(w_k - \hat{w}_k) \\ y_k &\approx g(\hat{x}_k, \hat{v}_k) + C_k(x_k - \hat{x}_k) + G_k(v_k - \hat{v}_k) \end{aligned} \tag{4}$$

where

$$\begin{aligned} A_k &= \left.\frac{\partial f(x, u_k, \hat{w}_k)}{\partial x}\right|_{x=\hat{x}_k} & F_k &= \left.\frac{\partial f(\hat{x}_k, u_k, w)}{\partial w}\right|_{w=\hat{w}_k} \\ C_k &= \left.\frac{\partial g(x, \hat{v}_k)}{\partial x}\right|_{x=\hat{x}_k} & G_k &= \left.\frac{\partial g(\hat{x}_k, v)}{\partial v}\right|_{v=\hat{v}_k} \end{aligned} \tag{5}$$

$\hat{w}_k = E\{w_k\}$ and $\hat{v}_k = E\{v_k\}$ are respectively the expectations of process noise and measurement noise at time instant k.

Kalman filter has two distinct phases: *Predict* and *Update*. The first phase produces a prediction of the current state based on the previous one, and in *Update* step, the current measurement is utilized to refine the prediction to obtain a revised and hopefully more accurate estimate of the current state. In order to implement the EKF, the *Predict* phase is accomplished using the original nonlinear function, while in the *Update* phase, the Kalman filter gain and covariance matrix are calculated by the linearized equations. The process is summarized as follow:

Predict

$$\begin{aligned} \hat{x}_{k|k-1} &= f(\hat{x}_{k-1|k-1}, u_k, \hat{w}_k) \\ P_{k|k-1} &= A_k P_{k-1|k-1} A_k^T + F_k Q_k F_k^T \end{aligned} \tag{6}$$

Update

$$\begin{aligned} K_k &= P_{k|k-1} C_k^T (C_k P_{k|k-1} C_k^T + G_k R_k G_k^T)^{-1} \\ \hat{x}_{k|k} &= \hat{x}_{k|k-1} + K_k(y_k - g(\hat{x}_{k|k-1}, \hat{v}_k)) \\ P_{k|k} &= P_{k|k-1} - K_k C_k P_{k|k-1} \end{aligned} \tag{7}$$

where $\hat{x}_{k|k-1}$ is the prediction of the state at the k-th instant, using the past observations before k, and $\hat{x}_{k|k}$ is the estimate of this state after using the observation y_k. $P_{k|k-1}$ and $P_{k|k}$ are also defined in a similar way, as estimates of the corresponding covariance matrices.

III. METHODOLOGY

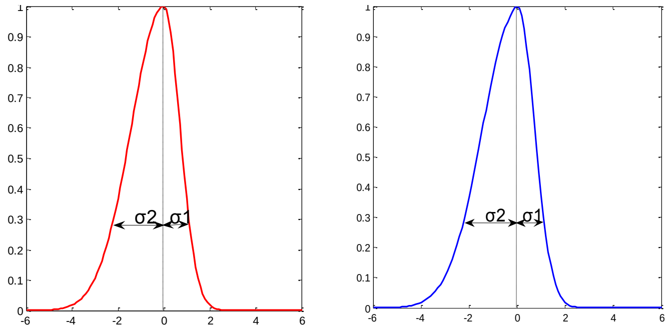
A. Generalization of EDM

If ECG feature waves (P, Q, R, S and T waves) can be assumed to have totally or nearly symmetric waveforms, the foregoing ECG dynamic model (EDM) can serve as a close approximation to the original signal. However, in reality, the waveforms of ECG signals are often not symmetric. Thus, it's difficult for the traditional EDM to accurately model them using symmetric Gaussians, which will further affect the performance of the model-based nonlinear filtering. Therefore, it's necessary to generalize the EDM to represent a wider range of ECG morphologies.

As an extension of conventional Gaussian function, asymmetric Gaussian (AG) can capture spatially asymmetric distribution well [6]. Traditionally, an AG is defined as:

$$\begin{aligned} &AG(x, a, \mu, \sigma_1, \sigma_2) \\ &= a \cdot \exp\left(-\frac{(x-\mu)^2}{2\sigma_1^2}\right) \cdot \chi(x-\mu) \\ &+ a \cdot \exp\left(-\frac{(x-\mu)^2}{2\sigma_2^2}\right) \cdot \left(1 - \chi(x-\mu)\right) \end{aligned} \tag{8}$$

where $\chi(u) = \begin{cases} 0 & u < 0 \\ 1 & u \geq 0 \end{cases}$ is a unit step function.



(a) (b)

Figure 2. Asymmetric Gaussians: (a) with the step function, and (b) with the sigmoid function

As shown in Fig. 2 (a), it is the difference between σ_1 and σ_2 that determines the asymmetry degree: the larger the difference, the more asymmetric the waveform.

However, the AG defined in (8) is not guaranteed to be derivable everywhere because of the step function. In order to apply AG to the EKF denoising framework, it's necessary to make it derivable throughout. Therefore, we propose to replace the unit step function with a sigmoid function as follows:

$$\begin{aligned} &AG(x,a,\mu,\sigma_1,\sigma_2) \\ &= a\cdot\exp\left(-\frac{(x-\mu)^2}{2\sigma_1^2}\right)\cdot\psi(x-\mu) \\ &+a\cdot\exp\left(-\frac{(x-\mu)^2}{2\sigma_2^2}\right)\cdot(1-\psi(x-\mu)) \end{aligned} \quad (9)$$

where $\psi(u)=1/(1+e^{-pu})$ is a sigmoid function with a parameter p which determines the function shape: as $p\to+\infty$, the sigmoid function evolves into a step function. Assigning a large enough value to p (e.g. p =5 in this paper), the AG defined in (9) is not only derivable everywhere now, but also has almost the same shape as that in (8), as shown in Fig. 2 (b).

With the proposed AG, the EDM in (1) can be generalized to be

$$ECG(\theta)=\sum_{i\in\{P,Q,R,S,T\}} AG(\theta,a_i,\theta_i,b_{i1},b_{i2}) \quad (10)$$

and the corresponding discrete form becomes:

$$\begin{aligned} &ECG(k+1)-ECG(k) \\ &=\delta_\theta\times\left.\frac{dECG(\theta)}{d\theta}\right|_{\theta=\theta(k)} \\ &=\delta_\theta\times\sum_{i\in\{P,Q,R,S,T\}}\left.\frac{dAG(\theta,a_i,\theta_i,b_{i1},b_{i2})}{d\theta}\right|_{\theta=\theta(k)} \end{aligned} \quad (11)$$

Since the AG parameters $a_i,\theta_i,b_{i1},b_{i2}$ vary from beat to beat due to the variation of heartbeat morphology, they should be included in the state vector. Thus, in the generalized EDM, the state vector consists of the 20 AG parameters, and the instantaneous ECG value at each time instant: $x=[a_P,\cdots,\theta_P,\cdots,b_{P1},\cdots,b_{P2},\cdots,ECG]^T$, the input is $u_k=\theta(k)$, and the output is $y_k=ECG(k)$. Then the nonlinear state space equations are then formulated as below:

$$\begin{bmatrix} a_P(k+1) \\ \vdots \\ \theta_P(k+1) \\ \vdots \\ b_{P1}(k+1) \\ \vdots \\ b_{P2}(k+1) \\ \vdots \\ ECG(k+1) \end{bmatrix} = \begin{bmatrix} a_P(k)+w_1(k) \\ \vdots \\ \theta_P(k)+w_6(k) \\ \vdots \\ b_{P1}(k)+w_{11}(k) \\ \vdots \\ b_{P2}(k)+w_{16}(k) \\ \vdots \\ f_{21}(ECG(k),u_k,w_{21}(k)) \end{bmatrix} \quad (12)$$

where $f_{21}(ECG(k),u_k,w_{21}(k))$ can be simply derived from (11) as follows:

$$\begin{aligned} &f_{21}(ECG(k),u_k,w_{21}(k)) \\ &=\delta_\theta\times\sum_{i\in\{P,Q,R,S,T\}}\left.\frac{dAG(\theta,a_i,\theta_i,b_{i1},b_{i2})}{d\theta}\right|_{\theta=\theta(k)}+ECG(k)+w_{21}(k) \\ &=\sum_{i\in\{P,Q,R,S,T\}} a_i\left(e^{-\frac{\Delta\theta_i(k)^2}{2b_{i1}^2}}\frac{e^{p\Delta\theta_i(k)}(pb_{i1}^2-\Delta\theta_i(k))-\Delta\theta_i(k)}{(1+e^{-p\Delta\theta_i(k)})^2 b_{i1}^2}\right. \\ &\qquad\left.-e^{-\frac{\Delta\theta_i(k)^2}{2b_{i2}^2}}\frac{e^{p\Delta\theta_i(k)}(pb_{i2}^2+\Delta\theta_i(k))+\Delta\theta_i(k)e^{-2p\Delta\theta_i(k)}}{(1+e^{-p\Delta\theta_i(k)})^2 b_{i2}^2}\right)\times\delta_\theta \\ &\quad+ECG(k)+w_{21}(k) \end{aligned} \quad (13)$$

where $\Delta\theta_i(k)=u_k-\theta_i=\theta(k)-\theta_i$.

Since the only observable state is the ECG signal, the observation equation is:

$$y_k=[0,\cdots,0,1]_{1\times21}x_k+v_k \quad (14)$$

B. *Linearization of the Nonlinear EDM*

The basic idea of EKF is to linearize the state space model at each time instant around the most recent state estimation. Given the generalized nonlinear EDM, the linearization matrices in (5) are calculated as below:

$$\begin{aligned} A_k &= \begin{bmatrix} I_{20\times20} & 0_{20\times1} \\ \frac{\partial f_{21}}{\partial a_P(k)},\cdots,\frac{\partial f_{21}}{\partial b_{T2}(k)} & 1 \end{bmatrix} \\ F_k &= I_{21\times21} \\ C_k &= [0,\cdots,0,1]_{1\times21} \\ G_k &= 1 \end{aligned} \quad (15)$$

C. *Initialization of EDM*

Before starting the EKF iteration process of (6) and (7), it's necessary to initialize the state vector x_0 first. An intuitive way is to estimate the initial state by experience. Although this method may be applicable to normal ECG signals, it's usually hard to estimate abnormal ECG features empirically.

A preferable way is to initialize the state vector by finding the optimal AG parameters that can best fit the AG to each beat of the ECG signal. As suggested in [7], by using a nonlinear least-square approach, the optimal estimate of the parameters in the MMSE sense can be found. In this paper, the *lsqnonlin* function in Matlab is utilized to implement the nonlinear optimization, and then initialize the state vector with the resultant parameters for the experiments presented later.

It's worth noting that it would be helpful to initialize the optimization process (*lsqnonlin* in this case) based on physiological knowledge, since the method tends to stop at a local optimal solution. Although abnormal heartbeats often have various waveforms, for example, the T wave may be inverted, the temporal order and distribution of all the waves in each beat are generally steady, which is helpful to the initialization of the optimization process.

IV. EXPERIMENT RESULTS

In this section, we carry out two experiments, respectively on ECG approximation and model-based denoising as follows.

A. *ECG Approximation*

In order to show the advantage in modeling complicated ECG morphologies, the generalized EDM is utilized to approximate both clean and noisy ECG signals.

The clean ECG signals are taken from the MIT-BIH Arrhythmia database [8], in which every record consists of two lead recordings sampled at 360 Hz with 11 bits per sample of resolution. The noisy signals are generated by corrupting the clean ones with white Gaussian noise, such that the Signal-to-Noise Ratio (SNR) gets to a specific value, e.g. 10 dB in this work.

For evaluating the approximating performance, we have adopted the sum of squared error (SSE) of each ECG beat defined by:

$$SSE = \sum_{i=1}^{N} (x(i) - \tilde{x}(i))^2$$

where x is the original signal (of one ECG cycle), and $\tilde{x}$ denotes the approximated signal obtained by model.

TABLE I. ECG APPROXIMATION RESULTS

Record \ SSE	Clean ECG		Noisy ECG	
	EDM	***G-EDM***	***EDM***	***G-EDM***
219	0.1096	0.0701	0.8192	0.7863
220	0.0826	0.0683	0.4092	0.3756
221	0.1136	0.0923	0.7799	0.7351
228	0.2318	0.1903	0.9216	0.8905
Mean	**0.1344**	**0.1053**	**0.7325**	**0.6968**

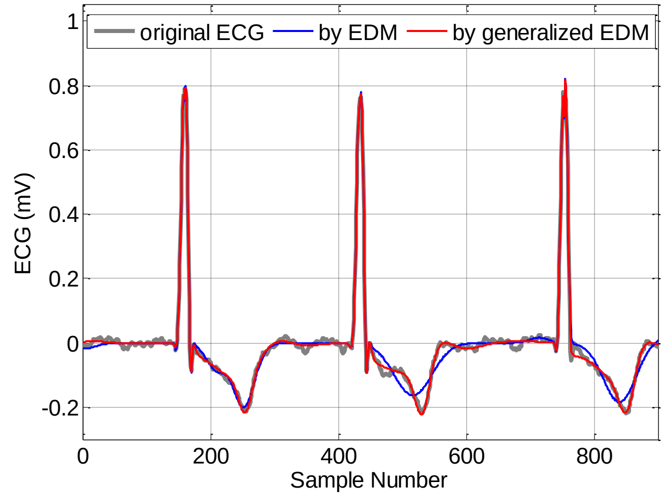


Figure 3. Clean ECG approximation by model

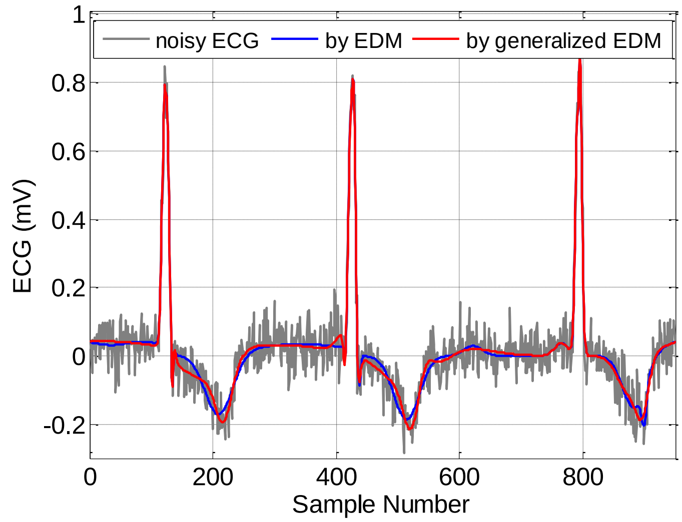


Figure 4. Noisy ECG approximation by model

Four records are selected from the database, each of which contains more than 2000 ECG cycles, and the average SSE of each record are calculated for comparison, as shown in Table I.

Fig. 3 has illustrated a sample portion of clean ECG signal, as well as the approximated signal obtained respectively by the traditional EDM and the generalized one, and Fig. 4 shows a sample portion of noisy ECG signal which is also approximated by the two methods.

From the numeric and figure comparisons above, we can conclude that due to the asymmetric waveforms in the original ECG, traditional EDM is not able to represent the morphology accurately, whereas the generalized model can approximate it well by virtue of the introduced asymmetric Gaussians.

B. *Model-based Denoising*

In order to quantitatively evaluate the denoising performance of the proposed method, we adopt the notion of SNR Improvement (*Imp*) defined by:

$$Imp[dB] = SNR_{output} - SNR_{input}$$
$$= 10\log\left(\frac{\sum_i |x_c(i)|^2}{\sum_i |x_d(i) - x_c(i)|^2}\right) - 10\log\left(\frac{\sum_i |x_c(i)|^2}{\sum_i |x_n(i) - x_c(i)|^2}\right)$$
$$= 10\log\left(\frac{\sum_i |x_n(i) - x_c(i)|^2}{\sum_i |x_d(i) - x_c(i)|^2}\right)$$

where x_c denotes the clean signal, x_n represents the noisy signal, x_d is the denoised one.

As shown in Table II, SNR improvements of noisy ECG inputs corrupted by different noises (5dB and 10dB) are calculated based on both denoising methods, and a sample of the denoising output is also illustrated in Fig. 5, where the noisy ECG has an SNR of 5dB. Both the quantitative and qualitative results demonstrate that the proposed method using generalized EDM has enhanced the ECG denoising performance.

TABLE II. EKF DENOISING RESULTS

Imp(dB) / Record	Input 10dB		Input 5dB	
	EDM	***G-EDM***	***EDM***	***G-EDM***
219	6.752	6.841	7.023	7.175
220	5.268	5.396	5.561	5.689
221	7.094	7.203	7.513	7.634
228	6.262	6.377	6.469	6.683
Mean	**6.344**	**6.454**	**6.642**	**6.795**

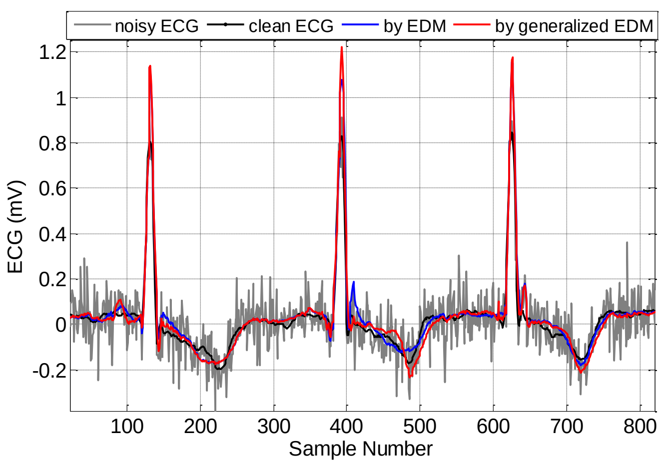


Figure 5. Model-based EKF denoising results

Conclusion and Future Work

In this paper, a modified version of asymmetric Gaussian is introduced by the use of sigmoid function instead of step function. Furthermore, the modified version is employed to generalize the ECG dynamic model (EDM) previously developed for generating synthetic ECG. The result is that the generalized EDM can accurately represent a larger variety of ECG morphologies, especially for those with asymmetric waveforms. Moreover, an extended Kalman filter (EKF) based framework for ECG denoising is implemented using the generalized EDM. Experimental results demonstrate that the denoising performance of our method has been improved by virtue of the enhanced ECG approximation ability of the generalized EDM.

Since the generalized EDM is capable of modeling the ECG feature waves more accurately, it may find applications in ECG feature extraction and classification, which is one of our future studies.

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